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COMMENT

Conformal charge of constrained fermionic models in a non-trivial topological background

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Abstract. We consider a constrained fermionic model in a non-trivial topological sector and compute the conformal charge showing its dependence on the topological charge.

Dotsenko and Fateev [1] have shown that the correlators of the general conformal theory in two dimensions can be represented by averages of vertex operators in a Coulomb-like system with non-trivial boundary conditions.

In their approach, in order to obtain minimal models, an appropriate background charge at infinity is added to an originally free boson system. This results in a modified boundary condition on the bosonic fields at infinity and then the holomorphic energy-momentum (EM) tensor gets an additional term:

$$\Delta T(z) = \mathrm{i}\alpha_0 \frac{\partial^2}{\partial z^2}\varphi(z) \tag{1}$$

where $2\alpha_0$ is the charge placed at infinity and $\varphi(z)$ is the bosonic field. The corresponding central charge is changed to:

$$C = 1 - 24\alpha_0^2. \tag{2}$$

Since the EM tensor has an imaginary part the theory it defines is non-unitary for arbitrary α_0, but with an adequate election of the charge one can obtain models in the minimal series.

Non-trivial boundary conditions can be naturally associated with topological structure. It is the aim of this work to investigate whether a generalization of the coset construction in fermionic models including topological effects leads to modifications of the EM tensor (and consequently of the central charge) analogous to those obtained by Dotsenko and Fateev [1]. To this end we shall follow a recent proposal by Bardacki and Crescimano [2] for treating constraints represented by gauge fields with non-trivial topology.

The coset construction can be realized in the path-integral framework starting from free fermionic models in which suitable currents are constrained by introduction of the gauge fields acting as Lagrange multipliers [3, 4]. We shall show that if one allows these gauge fields to have non-trivial topology one obtains a modification of the EM tensor resembling that in (1) with the topological charge playing the role of Dotsenko and Fateev's charge at infinity.

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We start from a fermionic Lagrangian in two-dimensional Euclidean spacetime coupled minimally to an Abelian gauge field A_μ which acts as a Lagrange multiplier (for details see [4]):

$$\mathcal{L} = \bar{\psi}(\mathrm{i}\not{\partial} + \not{A})\psi. \tag{3}$$

Dirac fermions ψ transform into the fundamental representation of $\mathrm{U}(k)$.

Consider the gauge field in the N topological charge sector, i.e.

$$\frac{1}{4\pi}\int_{S^2} {}^*F\,\mathrm{d}^2x = N \tag{4}$$

where S^2 is the two-dimensional sphere and ${}^*F$ is the dual of the electromagnetic stress tensor $F_{\mu\nu}$

$${}^*F = \varepsilon_{\mu\nu}F_{\mu\nu} = 2\varepsilon_{\mu\nu}\partial_\mu A_\nu. \tag{5}$$

The corresponding generating functional reads

$$Z_N = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}A^N \exp\left[-\int \bar{\psi}(\mathrm{i}\not{\partial} + \not{A}^N)\psi\,\mathrm{d}^2x\right] \tag{6}$$

where the A_μ^N integration is restricted to the N topological charge sector.

This theory is manifestly gauge invariant, so the gauge has to be fixed. We choose the condition

$$\partial_\mu A_\mu = 0. \tag{7}$$

The corresponding Fadeev–Popov determinant can be exponentiated using anticommuting ghost η and $\bar{\eta}$. One then has

$$\Delta_{\mathrm{FP}} = \int \mathcal{D}\eta\mathcal{D}\bar{\eta}\exp\left[-\int \eta\Box\bar{\eta}\,\mathrm{d}^2x\right]. \tag{8}$$

Two-dimensional fermionic models coupled to gauge fields are usually solved by performing a change of variables which decouple fermions from gauge fields [5]. However, for regular transformations this decoupling is valid only in the zero topological charge sector since for fields A_μ^N satisfying (4) it is not possible to globally define A_μ in the form

$$A_\mu^N = \varepsilon_{\mu\nu}\partial_\nu\phi \tag{9}$$

and this relation is at the root of the decoupling. One possibility to attack this problem is to allow for non-regular transformations leading to the fermion decoupling. Instead, following [2], we shall write A_μ^N in the form:

$$A_\mu^N = \tilde{A}_\mu^N + \varepsilon_{\mu\nu}\partial_\nu\phi \tag{10}$$

where $\tilde{A}_\mu^N$ is a fixed, patch-dependent configuration with topological charge N and the field ϕ can be defined globally since we take it in the zero topological sector.

Writing A_μ in this form, the ϕ-dependent part can be decoupled from fermions using the following change of fermionic variables [6]:

$$\begin{aligned}\psi(x) &= \exp(\gamma_5\phi(x))\chi(x)\\ \bar{\psi}(x) &= \bar{\chi}(x)\exp(\gamma_5\phi(x))\\ \mathcal{D}\bar{\psi}\mathcal{D}\psi &= J_{\mathrm{F}}\mathcal{D}\bar{\chi}\mathcal{D}\chi\end{aligned} \tag{11}$$

The fermionic Jacobian J_F can be evaluated using standard methods [5]. The answer is:

$$J_F = \exp\left[-\frac{k}{2\pi}\int [(\partial_\mu\phi)^2 - 2\phi\varepsilon_{\mu\nu}\partial_\mu\tilde{A}_\nu^N]\,d^2x\right]. \tag{12}$$

In terms of the new variables $\bar{\chi}$, χ and ϕ the generating functional reads

$$Z_N = \int \mathscr{D}\bar{\chi}\mathscr{D}\chi\mathscr{D}\phi\Delta_{FP}\times\exp\left[-\int\left(\bar{\chi}(i\not{\partial}+\tilde{\not{A}}^N)\chi - \frac{k}{2\pi}(\partial_\mu\phi)^2 - \frac{k}{2\pi}\phi{}^*\tilde{F}\right)d^2x\right] \tag{13}$$

where ${}^*\tilde{F} = \varepsilon_{\mu\nu}\partial_\mu\tilde{A}_\nu^N$.

We can now compute the Virasoro central charge of this model. In order to obtain the conformal anomaly we must study the non-regular term on the operator product expansion (OPE) of the EM tensor with itself.

It must be stressed that because of the non-zero topological charge of the field $\tilde{A}_\mu^N$, the index theorem [7] guarantees the existence of $k|N|$ square integrable solutions of the Dirac equation:

$$[i\not{\partial}+\tilde{\not{A}}^N]\eta = 0. \tag{14}$$

These zero modes can be evaluated explicitly [2] and one finds that for $N>0$ ($N<0$) the solutions are all right handed (left handed):

$$\eta_\alpha^a = \begin{pmatrix}\eta_{Ra}^\alpha\\ 0\end{pmatrix} \qquad N>0$$
$$\eta_\alpha^a = \begin{pmatrix}0\\ \eta_{La}^\alpha\end{pmatrix} \qquad N<0 \tag{15}$$

for $a = 1, 2, \ldots, |N|$ and $\alpha = 1, 2, \ldots, k$.

It is for this reason that any correlator of an arbitrary number of operators of the form $\bar{\chi}_R\mathscr{A}\chi_L$ (where $\mathscr{A}$ is some operator) vanishes. (An identical result is obtained with operators of the form $\bar{\chi}_L\mathscr{A}\chi_R$.)

Hence we conclude that we have no fermionic contribution to the total central charge. The fermionic part of the generating functional only appears as a multiplicative factor which cancels out when computing EM tensor vacuum expectation values. (Actually, due to zero modes this factor is zero and has then to be regularized, for example in the form proposed in [8].)

To obtain the bosonic contribution to the Virasoro anomaly we must compute the following correlation function:

$$\langle T_B T_B\rangle = \frac{1}{Z_B}\int \mathscr{D}\phi T_B(z)T_B(w)\exp[-S_B] \tag{16}$$

where Z_B is the bosonic part of the generating functional:

$$Z_B = \int \mathscr{D}\phi\,\exp[-S_B]. \tag{17}$$

S_B is given by:

$$S_B = -\frac{k}{2\pi}\int [(\partial_\mu\phi)^2 + \phi{}^*\tilde{F}]\,d^2x \tag{18}$$

and T_B in the right-hand side of (16) is the EM tensor associated with S_B.

It is useful to write the field $\tilde{A}^N_\mu$ in the form:

$$\tilde{A}^N_\mu = N\tilde{A}^1_\mu \tag{19}$$

where the field $\tilde{A}^1_\mu$ has topological charge equal to 1. Then writing:

$$\tilde{A}^1_\mu = \varepsilon_{\mu\nu}\partial_\nu\eta \tag{20}$$

the action takes the form:

$$S_{\mathrm{B}} = -\frac{k}{2\pi}\int [(\partial_\mu\phi)^2 - 2N\phi\Box\eta]\,\mathrm{d}^2x. \tag{21}$$

This action can be written as that of massless bosons interacting with a conveniently chosen gravitational field. To see this, consider (in S^2) the action [9]:

$$S = -\frac{k}{2\pi}\int \mathrm{d}^2x\,\sqrt{g}\,[g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - N\phi R] \tag{22}$$

where R is the scalar curvature.

We choose the conformally flat metric:

$$g_{\mu\nu} = \delta_{\mu\nu}\exp(4\eta) \tag{23}$$

where the field η is the same that appears in (20). In this metric we have

$$\sqrt{g}\,R = 2\partial_\mu\partial_\mu\eta. \tag{24}$$

It is important to stress that for this election of the gravitational field we have consistency with the Gauss–Bonnet theorem. This theorem can be seen as a topological constraint on the curvature of the surface of a given genus:

$$\frac{1}{2\pi}\int_{\mathscr{C}}\sqrt{g}\,R\,\mathrm{d}^2x = \chi(\mathscr{C}). \tag{25}$$

Here, $\chi(\mathscr{C})$ is the Euler characteristic:

$$\chi(\mathscr{C}) = 2g_{\mathscr{C}} - 2 \tag{26}$$

and $g_{\mathscr{C}}$ is the genus of the compact surface $\mathscr{C}$. For the sphere S^2, we have $g_{s^2} = 0$ and $\chi(s^2) = -2$.

In terms of the field η, the Gauss–Bonnet theorem reads:

$$\int_{s^2}\partial_\mu\partial_\mu\eta\,\mathrm{d}^2x = -2\pi \tag{27}$$

and this is nothing but the quantization condition (4) for the field $\tilde{A}^1_\mu$ (see (20)).

Then choosing the conformally flat metric (23) the action (22) becomes identical to that in equal (18).

The holomorphic EM tensor for the theory defined by the action (19) can be evaluated varying the action with respect to the metric:

$$T_{\mathrm{B}}(z) = -\tfrac{1}{2}:\partial_z\phi\partial_z\phi: + \tfrac{1}{2}N\partial_z^2\phi. \tag{28}$$

The additional (second) term in $T(z)$ comes from the interaction with the scalar curvature and has the same form as the correction found by Dotsenko and Fateev [1] (see (1)].

Note that our additional term is real while its counterpart in (1) is imaginary.

The contribution of the bosonic sector to the central charge can be easily calculated using (16):

$$\langle T_B(z) T_B(w)\rangle = \tfrac{1}{2}(1+3N^2)\frac{1}{(z-w)^4} \tag{29}$$

thus giving:

$$C_B = 1+3N^2. \tag{30}$$

In order to obtain the total central charge c_T one has to include the ghost contribution:

$$C_{\text{ghosts}} = -2. \tag{31}$$

Then the total central charge in the N topological charge sector is:

$$C_T = 3N^2 - 1 \tag{32}$$

Although we only deal with EM tensor correlators, all kind of multipoint correlation functions can be evaluated in our approach using the generating functional (13). One can then follow the programme of Belavin *et al* [10] for general conformal field theory.

It is interesting to note that our result for the central extension (32) is valid for $N \neq 0$. The $n=0$ case has as special feature that the Dirac equation (14) has no zero modes. In this case free fermions do contribute to the total central charge and we then have, instead of (32), the result [4]

$$C_T = k - 1 \tag{33}$$

that corresponds to an U(k) free fermionic theory (with $c=k$) in which an U(1) current is constrained.

In summary, by considering a constrained fermionic model in a non-trivial topological sector we have obtained a modified central charge (30) which depends on the topological charge N. Our approach was inspired by the proposal of Dotsenko and Fateev [1] concerning conformal theories with non-trivial boundary conditions. However in our model one naturally obtains a real holomorphic EM tensor and hence the theory it defines is automatically unitary; moreover, the central charge gets increased and then we do not obtain minimal models. In order to obtain such models within our approach, fermionic models constrained by non-Abelian gauge fields with non-trivial topology should be considered. It should be also of interest to discuss in this context how the central charge is modified if one considers manifolds other than S^2. We hope to report on this issue elsewhere.

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